# Regularized ART with Gibbs Priors for Tomographic Image Reconstruction[1]

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## Abstract

The Algebraic Reconstruction Technique (ART), which is based on Kaczmarz's projection algorithm, is one of the most important tools for tomographic consistent image reconstruction. Moreover, in the inconsistent case, an extension of Kaczmarz's method (KERP, for short) has been obtained by one of the authors in a previous paper. But, although theoretically very general, this extension cannot always produce an enough accurate reconstruction. In this respect, we consider in the present paper a regularized version of KERP algorithm (RKERP, for short), which demonstrates a very weak susceptibility to noisy perturbations in the data. The regularization is achieved through a penalty term in a least-squares objective to which the Kaczmarz's method is applied. This term is expressed with a Gibbs prior that incorporates nearest neighbor interactions among adjacent pixels. A special attention is drawn to a quadratic clique energy function that makes the Gibbs prior equivalent to a Gaussian prior. Our results demonstrate a high efficiency of the regularized KERP algorithm with such a prior as regards to a quality of the reconstructed images and a computational cost. In the simulations, we used the data from borehole tomography in which the inversion is very ill-posed due to a limitation of an angular range of the projections.

## 1   Introduction

In many image processing techniques the aim is to find a possibly good approximation to the true solution of the problem that can be often defined in terms of a linear least-squares problem. In this paper, we are dealing with tomographic image reconstruction in which a discretized forward projection model forms a system of linear equations

$$Ax + n = b, \tag{1}$$

where $A \in I\!\!R^{M \times N}$ is the system matrix, $x \in I\!\!R^N$ an unknown image vector, $b \in I\!\!R^M$ a measurement vector and $n \in I\!\!R^M$ is a noise vector that accounts for all kind of perturbations

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(measurement and discretization errors) which obviously make this system inconsistent, i.e. $b \notin R(A)$, where $R(A)$ is a range of $A$. Due to the intrinsic model of tomographic observations, $A$ is usually very ill-conditioned or rank-deficient, and such image reconstruction is in its nature very ill-posed. In our tomographic technique, $A$ is rank-deffcient even if we have more observations than pixels in the image, i.e. $M > N$. Assuming the linear least-squares problem: find $x^* \in I\!\!R^N$ such that

$$||Ax^* - b|| = \min_{x \in I\!\!R^N} ||Ax - b||, \tag{2}$$

its minimal norm solution, which we denote by $x_{LS}$, will be different that the true one (we shall denote by $|| \cdot ||, \langle \cdot, \cdot \rangle$ the Euclidean norm and scalar product, respectively). This is because $N(A)$ (the nullspace of $A$) is non-trivial, and some true image components that belong to $N(A)$ cannot be recovered from the observations (see e.g. [8]).

In this paper we consider the regularized linear least-squares problem in application to tomographic image reconstruction. The regularization term is not only to stabilize the solution of an ill-posed problem but, this term is to enforce a local smoothness in the image, and thus, it is modelled by a prior (like in statistical methods). We assumed the Gibbs prior associated with the MRF model. This approach is widely used in Bayesian image processing methods, e.g. MAP-EM [2, 3, 4, 6], which are usually classified as simultaneous techniques. However, in some applications of tomography the data are gathered sequentially, and sequential image reconstruction techniques like ART-like algorithms are more suitable.

## 2   The regularized KERP algorithm

The KERP algorithm, introduced in [7] can be written as follows.
**Algorithm KERP.** Let $x^0 \in I\!\!R^N, y^0 = b$; for $k = 0, 1 \ldots$ do

$$y^{k+1} = \Phi(\alpha; y^k); \; b^{k+1} = b - y^{k+1}; \; x^{k+1} = F(\omega; b^{k+1}; x^k). \tag{3}$$

Here $\alpha, \omega$ are relaxation parameters and the applications involved in (3) are defined by (see [7] for details)

$$f_i(\omega; b; x) = (1 - \omega)x + \omega f_i(b; x), \; F(\omega; b; x) = (f_1 \circ \cdots \circ f_M)(\omega; b; x), \tag{4}$$

$$\varphi_j(\alpha; y) = (1 - \alpha)y + \alpha\varphi_j(y), \; \Phi(\alpha; y) = (\varphi_1 \circ \cdots \circ \varphi_N)(\alpha; y), \tag{5}$$

$$f_i(b; x) = x - \frac{\langle x, a_i \rangle - b_i}{||a_i||^2} a_i, \; \varphi_j(y) = y - \frac{\langle y, \alpha_j \rangle}{||\alpha_j||^2} \alpha_j, \tag{6}$$

where by $a_i \in I\!\!R^n, \alpha_j \in I\!\!R^m$ we denoted the $i$-th row and $j$-th column of $A$, respectively (which we suppose to be nonzero vectors). We have the following result (see [7]).

**Theorem 1** *For any $x^0 \in I\!\!R^N$ and any $\omega, \alpha \in (0, 2)$, the sequence $(x^k)_{k \geq 0}$ generated by the algorithm* KERP *converges always to a least-squares solution of the problem* (2). *Moreover, for $x^0 = 0$ the limit is exactly its the minimal norm solution $x_{LS}$.*

**Remark 1** *The above* KERP *can eliminate in only one step the noise vector components from* $N(A^t)$ *(see [1]), but unfortunately this is not always enough in order to obtain a good image reconstruction (see e.g. [9]).*

In order to eliminate the above mentioned bad aspect we have to incorporate the prior information to the solution. This will be described in what follows. In this respect, we first consider the regularized weighted least-squares version of the problem (2): find $x^* \in R^N$ such that

$$\min \Psi(x^*) = \min_{x \in \mathbb{R}^N} \Psi(x), \quad \Psi(x) = ||Ax - b||^2_{\Sigma^{-1}} + \beta R(x), \tag{7}$$

where $\Sigma$ is a symmetric and positive definite $M \times M$ matrix which attributes weights to data, $\beta$ is a regularization parameter, and $R(x)$ is functional that measures the roughness in the image. Assuming the image modelled by the Markov Random Field (MRF) that is associated with the Gibbs prior

$$\pi(x) = \frac{\exp\{-\beta U(x)\}}{\int \exp\{-\beta U(x)\} dx}, \tag{8}$$

where $U(x)$ is a total energy function, and the data that are modelled by the Gaussian statistics, the discrete smoothing norm in (7) has the form: $R(x) = 2U(x)$. Then, following the same way as we proposed in [9], the regularized version of the above KERP algorithm can be written as follows.

**Algorithm RKERP.** Let $x^0 \in \mathbb{R}^N, y^0 = b$; for $k = 0, 1 \ldots$ do

$$y^{k+1} = \Phi(\alpha; y^k); \ b^{k+1} = b - y^{k+1}; \ x^{k+1} = F(\omega; b^{k+1}; x^k) - 2\beta \nabla U(x^k). \tag{9}$$

A common choice for $U(x)$ in (8) is the measure of a total roughness in the image, i.e.

$$U(x) = \sum_j \sum_{n \in N_j} w_{jn} V(x_j - x_n, \delta) \tag{10}$$

where $N_j$ denotes a set of the pixel indices from the nearest neighborhood of the $j$-th pixel, $w_{jn}$ is a weighting factor, and $V(x_j - x_n, \delta)$ is a clique energy function that is scaled with $\delta$. Many clique energy functions have been proposed to image reconstruction [2, 3, 4, 9]. Lange in [6] discussed their properties in the context of application to the MAP-EM algorithm. In our approach, we apply the quadratic function

$$V(x_j - x_n, \delta) = \left(\frac{x}{\delta}\right)^2, \tag{11}$$

which leads to considerable simplifications of the algorithm. Note that the Gaussian prior usually does not work well with the well-known MAP-EM algorithm due to oversmoothing. However, in our application we assume only the first-order interactions, which obviously decrease a local smoothness, and we use the ART-like algorithms. Considering this and (11) we get in (10)

$$U(x) = \frac{1}{\delta^2} x^T \left(I - \frac{W}{4}\right) x, \tag{12}$$

where $I \in I\!\!R^{N \times N}$ is the unit matrix, and

$$W = [w_{jn}] \in I\!\!R^{N \times N}, \qquad w_{jn} = \begin{cases} 1, & \text{for } \forall n \in \{N, E, W, S\}_j \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

where $\{N, E, W, S\}_j \subset N_j$ (see (10)) are the corresponding neighbours of the pixel $j$. Then, the appropriate step in the RKERP algorithm (9) becomes

$$x^{k+1} = F(\omega; b^{k+1}; x^k) - \gamma \left( I - \frac{W}{4} \right) x^k, \tag{14}$$

with $\gamma = \frac{4\beta}{\delta^2}$.

**Remark 2** *The Hessian of the minimization functional $\Psi(x)$ from (7) is given by $H = \nabla^2 \Psi(x) = A^T \Sigma^{-1} A + \frac{2\beta}{\delta^2} \left( I - \frac{W}{4} \right)$. By construction the matrix $I - \frac{W}{4}$ is symmetric and irreducible diagonally dominant, thus invertible. From its symmetry and Gershgorin's theorem it then result that it ia also positive definite. This, together with the positive definiteness of the matrix $\Sigma$ tells us that the Hessian $H$ is symmetric and positive definite, thus the functional $\Psi$ is strictly convex. This means that the regularized problem (7) has a unique solution which satisfy the "normal equation" $\nabla \Psi = 0$. This is an argument for considering the regularized Kaczmarz step (14) in (9) (see some details in [5]). But, unfortunately, we have not yet other systematic arguments for the convergence properties of the algorithm RKERP (the work in this direction is in progress).*

## 3 Numerical experiments

The tests presented here are performed on the same tomographic data as in [9]. The data are perturbed with a zero-mean Gaussian noise with $SNR = 30dB$. The true image and the minimal-norm least-squares solution and presented in Fig. 1 (left and right, respectively). Fig. 2 (top) shows the the images reconstructed with the **KERP** for 50, 150, and 250 iterations. The images obtained with the **RKERP** with the same numbers of iterations are shown in Fig. 2 (bottom) . The measures of distance and relative errors between the true image and the reconstructed image versus iterations are shown in Fig. 3. All the reconstructions are performed for optimally adjusted parameters $\omega = 0.6$ and $\alpha = 0.065$.
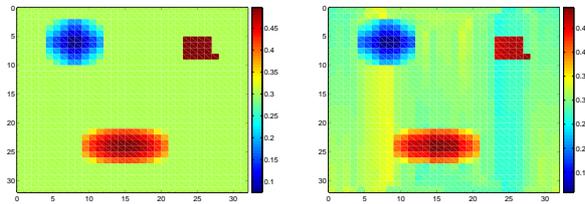


Figure 1: True image (left), and minimal-norm least-squares (right)
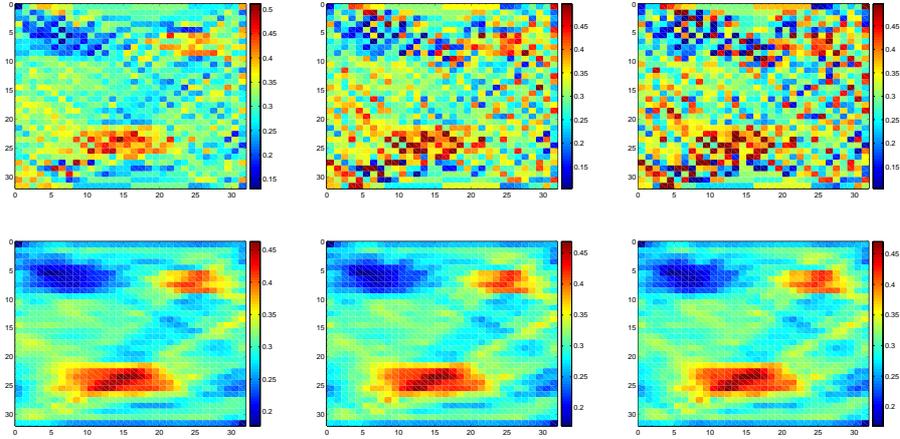
Figure 2: Image reconstructed with **KERP** (top) and **RKERP** (bottom) for $\gamma = 0.5$ within 50, 150 and 250 iterations, respectively
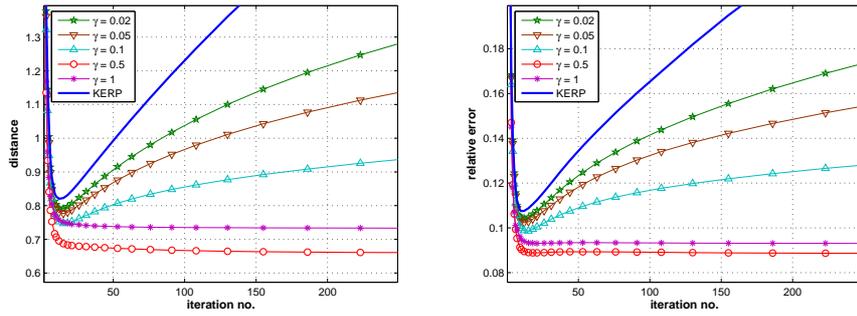


Figure 3: Distance (left) and relative errors (right) between the true image and $x^k$

## 4  Conclusions

The reconstructed images illustrated in Fig. 2 with the **RKERP** are very good approximations of the minimal-norm least-squares solution (Fig. 1 - right). The slight artifacts at the top and bottom result from the restriction of the MRF to the first-order interactions. Fig. 3 shows that the RKERP convergences monotonically, however, all the related parameters ($\alpha$, $\gamma$ and $\omega$) must be well adjusted. This can be also observed comparing the images reconstructed with the **KERP** and **RKERP** (Fig. 2).

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