A full multigrid technique to accelerate an ART scheme for tomographic image reconstruction

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Abstract

Tomographic reconstruction is the process of reconstructing a 3-D object or its cross section from several of its 2-D projection images. The object is illuminated by a cone-beam of X-rays, where the signal is attenuated by the object. Due to its speed filtered back projection (FBP) still is state-of-the-art in 3-D reconstruction for clinical use where time matters. But considering the accuracy and number of projections required for FBP, as shown in [1], an algebraic reconstruction technique (ART) is superior. Our current focus lies on 3-D angiography using C-arm systems. But this new approach should also be applicable on many real world reconstruction problems. Within ART, the object is represented as a linear combination of basis functions, typically voxels, with some unknown coefficients. The observations can also be expressed as a linear combination of these coefficients. This results in a linear system of equations with a sparse system matrix, because each X-ray intensity observation is influenced only by the pixels on the corresponding beam path. If enough measures are available, one has an over-determined system, which is solved in the least-squares sense. On the other hand, if there are not enough measures in a region to determine the coefficient values, one is faced with an under-determined problem. In this case, one solves the regularized version of the problem which supplies the additional constraints. Due to the large number of unknowns in real applications, an iterative instead of a direct linear solver has to be used. Techniques such as Kaczmarz's algorithm or CAV (component averaging) are currently used as iterative solvers, but for large problems, their computational costs are high. In addition, these solvers tend to improve the solution very much only in the first few iterations. An *efficient ART* is therefore essential to compete with FBP successfully.

In this paper we think of these iterative methods as smoothers within a *multigrid solver*. It should be noted that because of the structure of the system matrix, the standard multigrid

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theory is not applicable here. The additional ingredients of the multigrid method are coarser versions of the problem on different levels, interpolation and restriction operators. For the coarser problems, we uniformly reduce the number of rays and the number of voxels while keeping the overall volume constant. Furthermore, we use trilinear interpolation and full weighting as restriction. Full multigrid is then accomplished by starting on each level the V-cycle with an initial guess for the solution that is interpolated from the next coarser level. Our experiments show that we are able to reduce the relative error to a certain size by less Kaczmarz smoothing steps on the finest level when using the multigrid method instead of the common Kaczmarz algorithm. We present results for real medical datasets and compare our multigrid method with Kaczmarz and CAV on a phantom. One of the next steps will be to detail the theory for our multigrid method in order to get estimates for the asymptotic convergence rates.

1 Problem Description

Fast and accurate cone-beam reconstruction for X-ray computed tomography is still a challenging task. FBP is state-of-the-art in nowadays clinics because of its fast non-iterative solution scheme. But it is known that ART needs only one third of X-ray images compared to FBP [2] to reconstruct an image of comparable quality in 3-D. Many different algebraic reconstruction techniques like Kaczmarz (ART) [3], [4], Cimmino (SART), Censor and Gordon (CAV) [5], [6] where presented in the past (here we refer to the given articles for further details). The drawback of all iterative ART techniques is the complexity of the iterative formula applied on huge data sets. In practice, the reconstruction of a 256^3 or 512^3 volume from $P = 150$ X-ray images of 1024×1024 size is common. The practice shows that for a sufficient image quality a minimal number of about five iterations is necessary. To overcome this drawback, we introduce a full multigrid approach for the Kaczmarz algorithm. This allows to speed up the reconstruction, increase the accuracy by doing most of the work on coarser grids and only perform not more than three iterations on the finest grid. The crux of the cone-beam multigrid approach is to find a proper 2-D restriction and 3-D prolongation to alternate between different 3-D and 2-D grids.

In order to discretize the region of interest Ω, we introduce a Cartesian grid of cubes, called voxels, Ω^h that covers the whole volume that has to be reconstructed. We assume for simplicity that the length of each side of a voxel is h and denote the number of voxels by N. The X-ray attenuation function is assumed to take a constant uniform value x_j^h, for the voxel $j \in \Omega_j^h = \{1, 2, \dots, N\}$. We denote the number of rays in one projection by R, the number of projections by P and the number of rays in all projections by $M = RP$. The length of the intersection of the ith ray with the jth voxel is then d_{ij}^h for all $i \in \Omega_i^h = \{1, 2, \dots, M\}$ and $j \in \Omega_j^h$. a_{ij}^h therefore represents the contribution of the jth voxel to the total attenuation along the ith ray and is computed via alpha-clipping. The total attenuation along the ith ray is denoted by b_i^h, which represents the line integral of the unknown attenuation function along the path of the ray. Thus the discretized model can be written as a system of linear

equations

$$A^h x^h = b^h \ , \quad \sum_{j \in \Omega_j^h} a_{ij}^h x_j^h = b_i^h \ (i \in \Omega_i^h) \ . \tag{1}$$

We call $b^h \in \mathbb{R}^M$ the measurement vector, $x^h \in \mathbb{R}^N$ the image vector and $A^h \in \mathbb{R}^{M \times N}$ the projection matrix.

In order to solve the linear system (1), we cannot use a direct solver since A is very huge. Instead, we are using iterative solvers. A has the property that it is very sparse, since each of its rows contains the intersection length of one single ray that hits only a few voxels. In addition, all entries of A are non-negative and two neighboring ray equations are very similar because of a small angle between the rays. It should also be noted that (1) can be an over-determined or under-determined system, depending on the number of rays and projections. It does not have a unique solution in general. In such a case, we are looking for a least squares solution, i.e. we solve the system

$$(A^h)^T A^h x^h = (A^h)^T b^h \ , \tag{2}$$

shortly $\mathbf{A}^h x^h = \mathbf{b}^h$, instead.

2 The Multigrid Algorithm

Multigrid algorithms are known to be optimal in terms of computational costs for solving sparse linear systems (cf. [7], [8]). But they were orignally developed for elliptic PDEs and the design of a multigrid method for a new problem can be a difficult task, especially when the application is far away from the classical multigrid setting. Therefore we started with a standard multigrid method and tried to adapt it to equation (2) (cf. [9]).
Here, we analyse for simplicity only a two-grid multigrid. The recursive extension to a hierarchy of several grids is straightforward. We split Ω^h into two disjoint subsets $\Omega^h = C^h + F^h$, where C^h represents the variables contained in the coarse level and F^h is the complementary set. Given such a splitting and defining the coarse grid $\Omega^H = C^H \subset \Omega_h$ ($H = 2h$) we get the coarse-level system

$$\mathbf{A}^H x^H = \mathbf{b}^H \ . \tag{3}$$

The usual multigrid efficiency is achieved through the combination of two iterations, the smoother and the coarse grid correction. It should be mentioned that the construction of the coarse grid operator A_H is based on direct discretization since our problem is defined on a regular grid. That means we just skip every second ray on the coarser grid and bisect the number of voxels in every dimension while preserving the physical volume size by doubling the edge length of each voxel. Then, one multigrid V-cycle starts with one or more smoothing steps. We describe a smoothing step with the smoothing operator S_h:

$$x^h \rightarrow \bar{x}^h \ , \quad \bar{x}^h = S_h x^h + (I_h - S_h)\mathbf{A}_h^{-1} b^h \ .$$

We consider a Kaczmarz iteration ([3]) as a smoother. It uses $S_h = (I_h - (A_h)^T \lambda D^{-1} A_h)$, where

$$D^{-1} = diag \left(\frac{1}{\|a^1\|_2^2}, \frac{1}{\|a^2\|_2^2}, \dots, \frac{1}{\|a^M\|_2^2} \right) \ ,$$

with the rows $a^i, i \in \{1, 2, \dots, M\}$ of A^h. This can be thought of as a damped Jacobi smoother for equation (2).

With the exact solution x_*^h of (2) the error is defined by $e^h = x_*^h - x^h$. After the smoothing operation we solve on the coarse grid the following equation

$$\mathbf{A}^H e^H = \mathcal{I}_h^H r^h = \mathcal{I}_h^H (\mathbf{b}^h - \mathbf{A}^h \bar{x}^h) \ ,$$

by choosing both the 2-D restriction $\mathcal{I}_h^H$ and 3-D prolongation $\mathcal{I}_H^h$ as full weighting. Then the next step is the correction

$$x_{new}^h = \bar{x}^h + \mathcal{I}_H^h e^H \ ,$$

and again several smoothing steps after correction if necessary. The error equation on the coarse grid can now also be solved by a two-grid cycle recursively, which leads to the multigrid method. We denote the pre-smoothing steps by (SFC) and the post-smoothing steps by (SCF) of a V-cycle. In practise we use a maximum number of L levels and do not solve the error equation on the coarsest grid exactly, but perform only a fixed number of smoothing steps (denoted by SE). For full multigrid (FMG) we start at the coarsest level and compute there the solution. After that the solution is interpolated to the next finer grid and used there as initial solution for one or more V-cycles. This process continues up to the finest grid, where again one or more V-cycles are done.

3 Experimental Results

First, we evaluate the FMG (MG-Kaczmarz) on a phantom and second on real CT data ($N = 128^3$, $M = 512^2 \times 133$) (see Fig. 1). For the reconstruction we use the *digitally reconstructed radiographs (DRRs)* from the phantom x^{Ph} ($N = 64^3$, $M = 256^2 \times 133$). We compute the *relative error* $\epsilon^{(k)}$ of the reconstruction to the ground truth x^{Ph} with

$$\epsilon^{(k)} := (\sum_{j=1}^N |x_j^{Ph}|)^{-1} \sum_{j=1}^N |x_j^{(k)} - x_j^{Ph}|.$$

The phantom contains spheres with decreasing radius, aligned along a helical trajectory. The surrounding background describes soft tissue simulated by smoothed random intensity values. In Fig. 2 we compare the relative error $\epsilon^{(k)}$ of CAV, Kaczmarz and our new MG-Kaczmarz after k workunits. For CAV and Kaczmarz one workunit implicates the processing of all $P = 133$ projection images and for MG-Kaczmarz one V-cycle implicates the workunits of $SFC + SCF$ on the finest grid plus the work properly scaled on coarse grids (including restriction and prolongation). We noticed in our experiments that the relaxation λ is crucial. An unfavourable λ can lead to overshoots of the computed relaxation and the convergence will be very slow because after adding the error to the current solution the result will not improve. With a well chosen λ the multigrid approach can beat the traditional Kaczmarz, especially for large volume data.
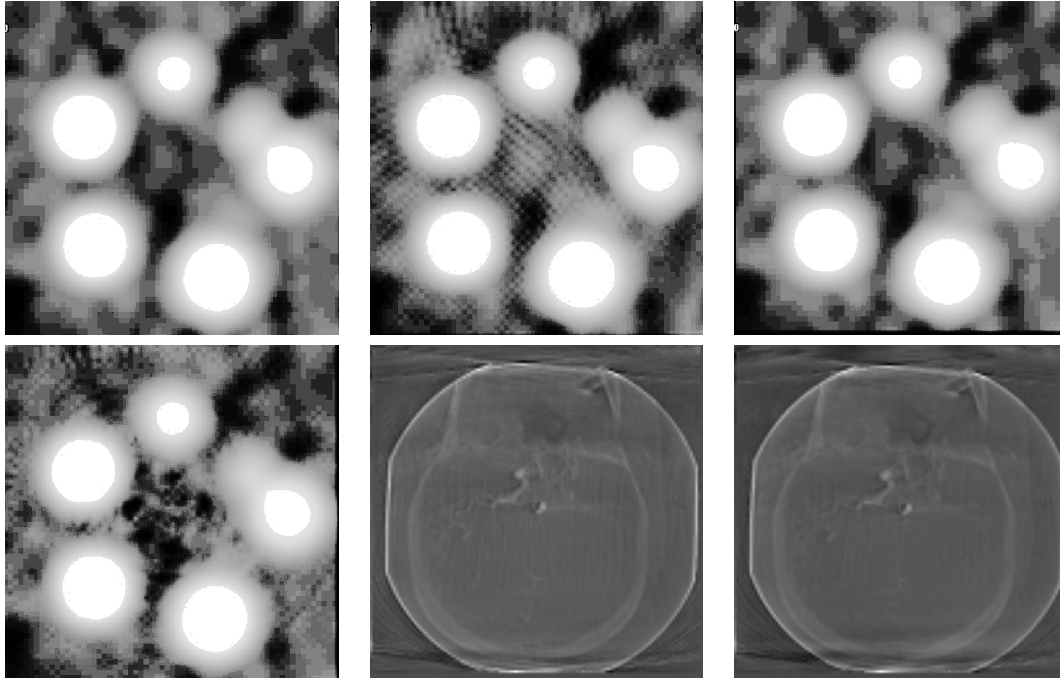
Figure 1: In the top row from left to right: original center slice (CS) of x^{Ph}; CS Kacz. (without MG after $k = 5$ iterations cone-artifacts are still visible); CS MG-Kacz. after one V-cycle shows sufficient image quality without strong cone-artifacts ($L = 4$, $SFC = 2$, $SE = 15$, $SCF = 0$, $\lambda = 0.05$); Bottom row: CS CAV ($\lambda = 1.0$, $k = 10$); Real CT head CS: Kaczmarz ($k = 5$, $P = 133$, $\lambda = 0.05$) with comparable image quality to MG-Kaczmarz (right) after one V-cycle ($L = 5$, $SFC = 3$, $SE = 20$, $SCF = 0$, $\lambda = 0.05$, $P = 133$).

4   Conclusion and Outlook

We have shown the gain from using a multigrid method for solving the image reconstruction problem. MG-Kaczmarz can beat both CAV and Kaczmarz after only a few workunits and still provides an acceptable image quality. Next steps will be a detailed analysis of the used solver and applying several techniques to improve the multigrid. For example one could think of using extended versions of the Kaczmarz smoother (cf. [6]) or of using Galerkin coarsening (cf. [7]) for constructing the coarse grid equations.

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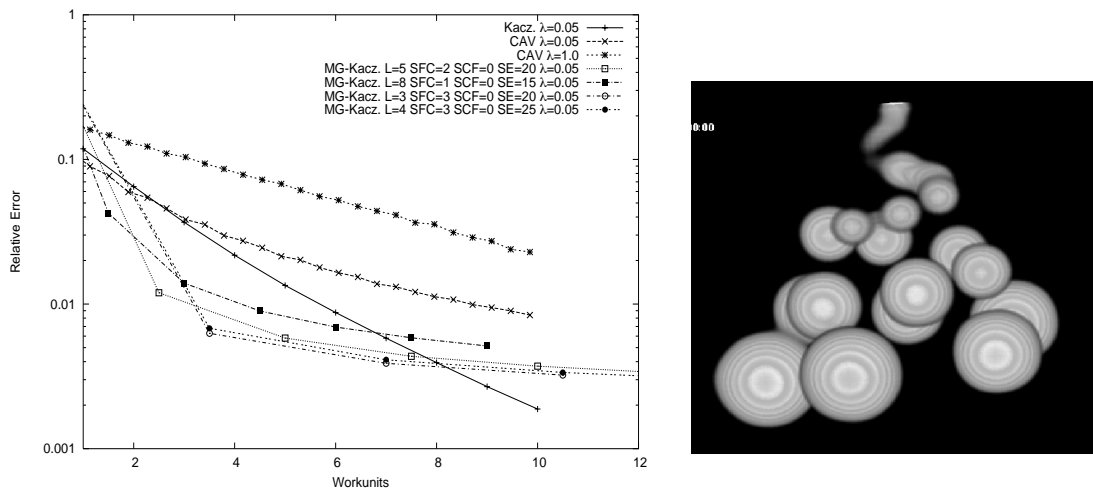
Figure 2: Left: Relative error $\epsilon^{(k)}$ of the reconstructed phantom. Kaczmarz becomes superior to CAV with increasing workunits k. After two workunits MG-Kaczmarz beats both CAV and Kaczmarz. The image quality with a reconstruction error of $\epsilon^{(k)} < 0.01$ is sufficient. Right: 3-D rendered phantom.

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