Efficient 3D Simulation of Wave Propagation with the Functional Transformation Method

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Abstract

The functional transformation method is a elegant mathematical approach for the solution of partial differential equations and has been so far successfully applied in digital sound synthesis for the simulation of string and percussion instruments. Based on suitable integral transformations, the problem is solved analytically in both, time and space frequency domain. Discretization and inverse transformation yields a discrete algorithm, implementable in any computer hardware. In this paper the functional transformation method is used for the simulation of three-dimensional (3D) wave propagation. With the help of a multi-dimensional fast Fourier transform, the complete wave field can be evaluated and visualized very efficiently. A program is demonstrated which simulates a 90m^3 room with an audio bandwidth of 5kHz at 5 images per second on current PC-hardware. The algorithm is compared with classical simulation techniques both in computational complexity and accuracy.

1 Introduction

The simulation of acoustical wave fields is a growing area of application, on one hand facilitated by the increasing computational power of modern hardware and on the other hand demanded by room acoustics research. Especially for the upcoming multi-channel reproduction techniques like wave field synthesis [BVV93], simulations are required to validate the output of the system and to ease further research.

Typical methods from literature for the simulation of acoustical wave fields are the finite difference time domain (FDTD) method [SRT94], waveguide meshes [LV00], and the mirror image method [FHLB99]. However, the first two methods are based on the spatial discretization of the modeled wave field, what yields undesired dispersion effects. The mirror image method is based on acoustical rays, what is only accurate for high frequencies. It is usually applied to achieve room impulse responses and is not suitable for the simulation of complete wave fields.

Therefore, in this paper a new approach is presented, where the functional transformation method (FTM) [TR03] is applied for the simulation of 3D acoustical wave fields. In doing so the mathematical model of wave propagation in terms of a partial differential equation (PDE) with suitable initial- and boundary- conditions is solved analytically

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in the frequency domain. Using simple geometries, which can be connected together to complex models with methods described in [PR05b], one can take advantage of highly efficient implementations via a three-dimensional FFT (see [PR05a] for the corresponding 2D implementation), yielding a fast and accurate tool for the simulation and visualization of acoustical wave fields.

The paper is organized as follows: in section 2 the simulation of acoustical wave fields with the FTM is described in brief. Referencing the existing 2D implementation [PR05a], some special aspects of the 3D implementation from this paper are given in detail. Section 3 presents the implementation of this algorithm in the program "*Wave3D*". Simulation results and a performance comparison with other methods are given. Section 4 concludes this paper.

2 Simulation of Wave Propagation with the Functional Transformation Method

The proposed method is a 3D-expansion of the 2D algorithm described in [PR05a]. Therefore only a brief overview is given here.

2.1 The Physical Model

The simulation of acoustical wave fields is based on the following physical principles (see [RF04] for instance) which relate the differential acoustic pressure $p(\vec{x},t)$ and the differential particle velocity $\vec{v}(\vec{x},t)$ by the first order PDEs

$$\begin{aligned} -\frac{\partial}{\partial t}p(\vec{x},t) &= \varrho_0 c^2 \nabla \vec{v}(\vec{x},t) && \text{equation of continuity} \\ -\nabla p(\vec{x},t) &= \varrho_0 \frac{\partial}{\partial t}\vec{v}(\vec{x},t) && \text{equation of motion}\ , \end{aligned} \tag{1}$$

where c is the speed of sound and ϱ_0 is the mass density of air. Both, the divergence of the particle velocity and the gradient of the pressure are described by the nabla-operator ∇, which is defined in three-dimensional Cartesian coordinates by $\nabla = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3}$ ($\vec{e}_1,\ \vec{e}_2$, and $\vec{e}_3$ are the Cartesian unit vectors). Both equations (1) can be easily combined to the first order vector PDE

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla + \begin{pmatrix} 0 & -1 \\ -\frac{1}{c^2} & 0 \end{pmatrix} \frac{\partial}{\partial t} \right] \underbrace{\begin{pmatrix} p(\vec{x},t) \\ -\varrho_0 \vec{v}(\vec{x},t) \end{pmatrix}}_{=\mathbf{y}(\vec{x},t)} = \begin{pmatrix} 0 \\ -f_\mathrm{e}(\vec{x},t) \end{pmatrix}\ , \tag{2}$$

with $f_\mathrm{e}(\vec{x},t)$ denoting an arbitrary excitation, a distribution of loudspeakers for instance. To complete the model initial conditions and boundary conditions are required. For simplicity initial conditions are assumed to be homogeneous, i.e. $\mathbf{y}(\vec{x},0) \equiv \mathbf{0}$. To define the boundary conditions first of all the model geometry is needed.

The FTM can handle any geometry, however it may turn out difficult to solve the eigenvalue problem needed for the integral transformation with respect to space (see [TR03]).

Furthermore, the efficient FFT implementations from [PR05a] require separable boundary conditions (i.e. rectangular regions in Cartesian coordinates or circular regions in polar coordinates). Therefore a different approach as described in [PR05b] is chosen: complex room models are broken into several sub-model with simple geometry. These models are solved separately with the FTM and are reconnected in the discrete system via the interaction of their boundary conditions.
Therefore, in this paper as a simple but nonetheless useful geometry, a cube with size $l_1 \times l_2 \times l_3$ is chosen. Furthermore the boundaries are assumed to be perfectly reflecting, i.e. the normal component of the particle velocity at the boundary is zero.

2.2 Application of the Functional Transformation Method

The FTM starts from the problem description derived in the previous section 2.1 and solves it analytically in the frequency domain with the help of several integral transformations. The Laplace transformation ($\mathcal{L}\{\cdot\}$) yields a PDE in space only, as all temporal derivatives are replaced by powers of the time-frequency variable s. The following problem specific Sturm-Liouville Transformation (SLT, $\mathcal{T}\{\cdot\}$) acts similar on the spatial derivatives, resulting in a transfer function, both in time and space frequency domain. Discretization with the impulse-invariant transformation, inverse $\mathcal{Z}$-transformation, and inverse SLT yield the desired discrete solution in terms of weighted complex first order recursive systems.

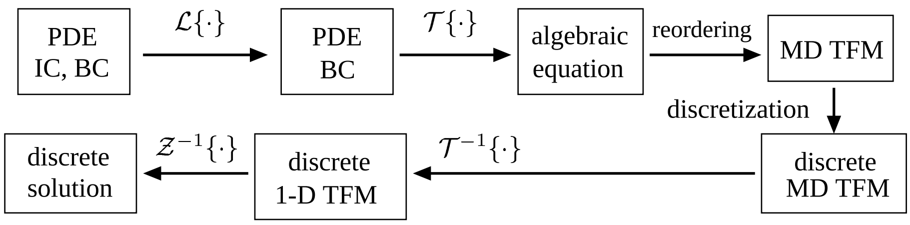

Figure 1: *General procedure of the FTM solving initial-boundary-value problems defined in form of PDEs and initial conditions (IC) and boundary conditions (BC). Further abbreviations are explained in the remainder of this section.*

The actual solution of the initial-boundary-value problem is performed by the search for the eigenvectors $\mathbf{K}(\beta, \vec{x})$ and the corresponding eigenvalues β of a Sturm-Liouville type problem, which are needed for the inverse SLT, resp. the search for the adjoint eigenvectors $\widetilde{\mathbf{K}}(\tilde{\beta}, \vec{x})$ and the corresponding adjoint eigenvalues $\tilde{\beta}$, which are needed for the SLT. In doing so, the eigenvectors can be determined by elementary matrix operations (see [TR03]), however the discrete values of β_μ (finite spatial regions always yield discrete eigenvalues) have to be found by an extensive search, especially for complex spatial regions.
Nethertheless, as a simple geometry was chosen in section 2.1, the eigenvalues β_μ can be given analytically in this scenario

$$\beta_\mu = j\omega_\mu := \pm jc\sqrt{\left(\frac{\mu_1 \pi}{l_1}\right)^2 + \left(\frac{\mu_2 \pi}{l_2}\right)^2 + \left(\frac{\mu_3 \pi}{l_3}\right)^2}\,, \tag{3}$$

with j denoting the imaginary unit and $\mu_1, \mu_2, \mu_3 \in \mathbb{N}_0$, each integer value corresponding to one spatial dimension.

2.3 Efficient Realization with the FFT

The computational effort for the FTM is mainly determined by the Nyquist frequency of the discretization. As only frequencies ω_μ (see equation (3)) below the Nyquist frequency $\frac{f_s}{2}$ (half the sampling frequency) have to be considered, one can determine the maximum number of the integers in equation (3) to be

$$N_i := \mu_i \,|_{\max} = \frac{l_i}{c} f_{\mathrm{s}} \qquad i \in \{1; 2; 3\} \,. \tag{4}$$

As each recursive system requires 3 floating point multiplications (FPM), one can estimate the total amount of FPM by $3N_1N_2N_3$. However, so far the inverse SLT has to applied for each evaluation point separately, i.e. a visualization of the complete wave field with $N_1 \times N_2 \times N_3$ evaluation points (optimal spatial resolution) would require $3\,(N_1N_2N_3)^2$ FPM. Fortunately, for this simple geometry and equally spaced evaluation points, one can prove the equivalence of the inverse SLT with a three-dimensional FFT, so that efficient FFT-algorithms (see [FJ98] for instance) can be applied. For cubic rooms with $N := N_1 = N_2 = N_3$ the number of FPM drops down from $3N^6$ to $N^3(3 + \log_2 N)$.

3 Implementation and Results

The implementation of the proposed algorithm and a performance comparison are given in this section.

3.1 The Program "*Wave3D*"

The proposed method is implemented in the program "*Wave3D*". It simulates wave propagation in a 3D rectangular room and displays a 2D cut of this wave field in the main window, as it can be seen in the screenshots in figure 2. It has a comfortable graphical user interface, where all parameters can be edited and saved, resp. loaded. In dependence on the chosen room-sizes and sampling frequencies, fluent animations of the wave field are created, for instance 5 frames per second for a 90m^3 room with a Nyquist frequency of 5kHz on current PC-hardware.

3.2 Performance

The comparison of the performance of the program "*Wave3D*" with methods from literature is carried out in two steps: first the program is compared, both in computational effort and accuracy, to a FDTD implementation at our laboratory. Then the absolute time per iteration (per sample) is compared with a waveguide mesh implementation from literature. The accuracy of the FDTD method and the waveguide mesh can be assumed to be similar. Further

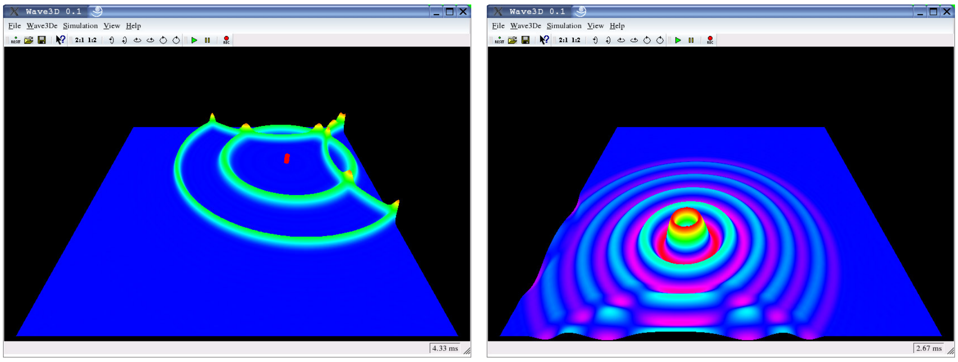

Figure 2: Screenshots taken from the program "*Wave3D*". It displays the 2D-cut from a 3D wave-field, excited by an impulse (left side) resp. by a sine-function (right side).

methods, as the finite element method or the mirror image method are not considered, as these methods are much more demanding in terms of computational power.
As a test scenario, the parameters from the waveguide implementation in [ME04] are chosen, i.e. a rectangular room of size $2\mathrm{m} \times 2\mathrm{m} \times 1\mathrm{m}$ and a spatial resolution of 2cm, resulting in a total number of $100 \times 100 \times 50 = 0.5 \cdot 10^6$ nodes. A FDTD implementation needs 8 FPM per node, resulting in $4 \cdot 10^6$ FPM per time step. The FTM implementation requires $(3 + 7) \cdot 0.5 \cdot 10^6 = 5 \cdot 10^6$ FPM per time step or 25% more.
The required time step size can be determined from the stability condition of the FDTD implementation (see [SRT94]). It results in a Nyquist frequency of 15kHz, while the Nyquist frequency of the FTM-implementation is only 8.7kHz (see equation (4)). Nevertheless, both simulations were run at a temporal sample rate of 30kHz, i.e. a sample step size of $T = 33\mu\mathrm{s}$. The excitation was a Gaussian impulse with a strong lowpass characteristic such that no frequency components above the Nyquist frequency were excited. The Gaussian curve was adjusted such that all frequencies above 15kHz were more than 80dB below the peak value.
The resulting simulations can be seen in figure 3. They show the response to a Gaussian pulse in the center 2.33ms after the excitation. The left plot was created with the program "*Wave3D*", the right plot was created with a FDTD implementation in MATLAB. It is obvious, that the results of the FDTD method suffer from dispersion, which is not present in the results of the FTM method.
For an exact comparison of the simulation accuracy, the results of both methods would have to be compared against an analytic result or a highly accurate numerical solution. A somewhat simpler approach was taken here. The step size of the FDTD solution has been decreased until no more dispersion was visible in the results. The FDTD simulations with increased spatial resolution (and in consequence increased temporal resolutions) are depicted in figure 4. These simulations require many more FPM, however still some dispersion artefacts are visible. The final FDTD simulation is depicted in figure 5 on the right.

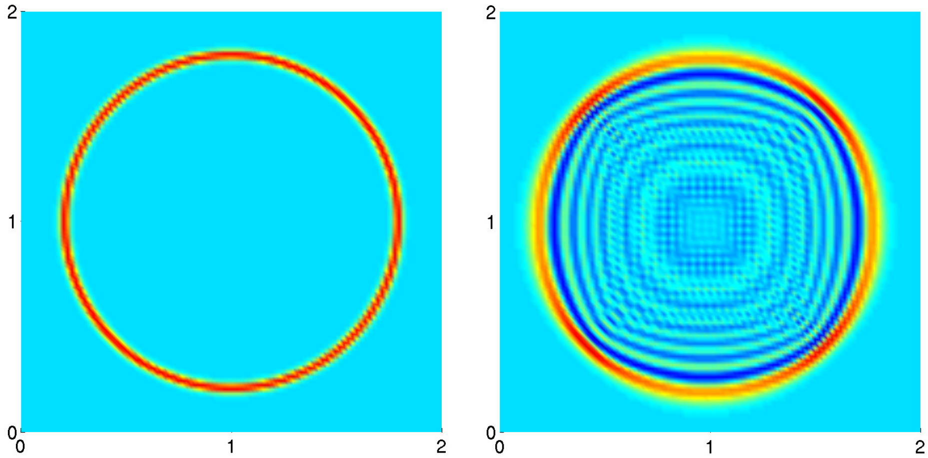

Figure 3: 2D snapshots of a 3D wave field simulation of a $2 \times 2 \times 1\mathrm{m}^3$ room at $f_\mathrm{s} = 30\mathrm{kHz}$. The left plot is simulated with the FTM and the right plot with the FDTD method. The dispersion artefacts of the FDTD simulation are clearly visible. The wave amplitudes are shown in a linear color scale.

numerical method	FTM	FDTD	FDTD
spatial step size	2 cm	2cm	0.5 cm
FPM per time interval T	$5 \cdot 10^6$	$4 \cdot 10^6$	10^9

Table 1: Number of floating point multiplications (FPM) per time interval $T = 33\mu\mathrm{s}$ for the results of the FTM and FDTD implementations shown in figure 3 and figure 5 .

The spatial and the temporal resolution is 4 times higher (i.e. 0.5cm and 8.33μs) than in figure 3, yielding an increase in the number of required FPM of $4^4 = 256$. On the left side of figure 5 the FTM simulation from figure 3 with interpolated points is shown. Both plots are in a logarithmic scale. The error of the FTM simulation is equally distributed and below -40dB, whereas the error of the FDTD simulation is concentrated shortly after the main slope and reaches about -30dB. The results of these simulations are compiled in table 1. They show that for comparable accuracy (see figure 5) the FTM method is faster than the FDTD method by a factor of almost 200.
A comparison with waveguide mesh implementations from literature gives similar results. The absolute time for one time step of the proposed scenario with "*Wave3D*" is about half a second on a Pentium III 1000MHz PC with 512Mb RAM. This is almost identical to the computation times measured in [ME04]. However, the waveguide mesh suffers from dispersion too, as it can be clearly seen in screenshots given in [BM04]. Trying to achieve the same accuracy with a waveguide mesh and the FTM implementation would yield similar results as above.

4 Conclusions

A new approach for the simulation of 3D acoustical wave fields has been presented. The acoustical wave equation was solved with the functional transformation method in the frequency domain. Fast FFT implementations were used to evaluate the entire wave field. A simulation tool called "*Wave3D*" was presented, that simulates wave propagation in a 3D rectangular room. The proposed approach does not suffer from dispersion. This advantage turned out clearly in performance comparisons with a FDTD implementation and a waveguide mesh implementation from literature.

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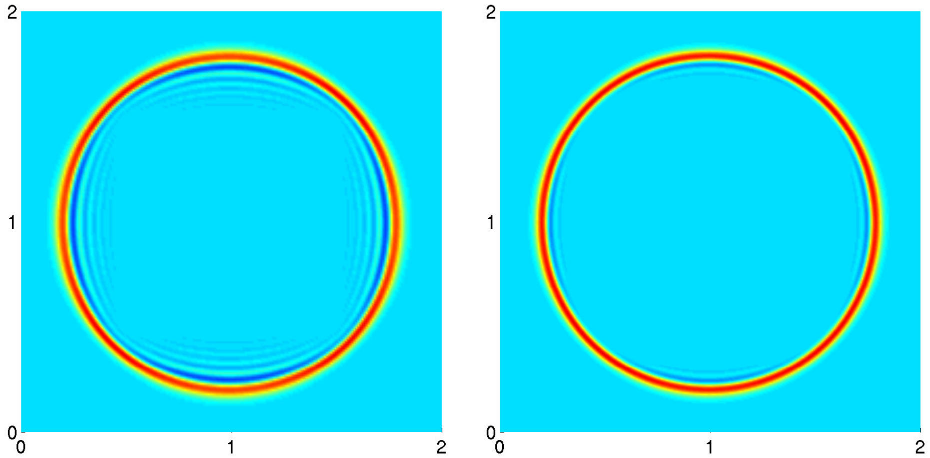

Figure 4: Snapshots of FDTD simulations with identical parameters as in figure 3. Only the spatial resolution was increased to 1cm (left plot), resp. 0.66cm (right plot). Dispersion artefact are still visible.

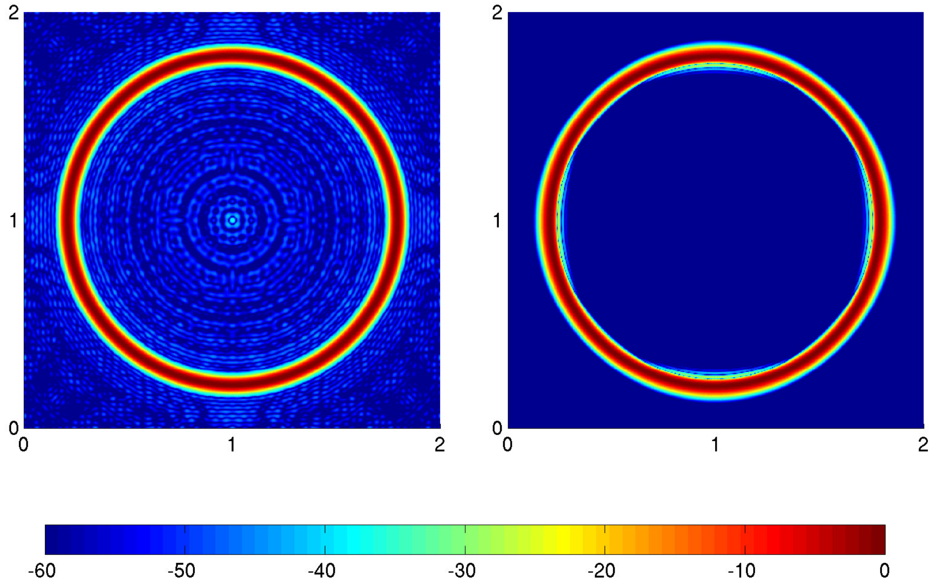

Figure 5: FTM (left plot) and FDTD (right plot) simulations in a logarithmic scale with parameters from figure 3, but a 4 times higher spatial resolution. The color bar is labeled in dB below peak value.